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Quantum Theory and Geometry: Sixty Years After von Neumann

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This paper is a brief review of some of the developments in the mathematical foundations of quantum mechanics that have taken place since the publication in 1932 of John von Neumann's celebrated treatise *Mathematische Grundlagen der Quantenmechanik*.

1. INTRODUCTION

It is a great honor for me to be able to address this first conference of the International Quantum Structures Association (IQSA). I would like to thank Prof. Beltrametti and the organizers of the conference for giving me this opportunity. The number and diversity of the scientific talks given during the past few days are an indication that the mathematical structure of quantum theory is still a very active and fruitful subject of study. I feel that the only way I can contribute to this conference is to give a general survey of some of the themes and results in this area. I also wish to take this opportunity to mention that it was through the lectures of Prof. G. W. Mackey at Seattle in 1961 on the mathematical foundations of quantum mechanics that I first began to appreciate the beauty and depth of the subject, and after 30 years it still retains its very great fascination for me.

It is of course not possible for me to give a thorough discussion of the main developments of this subject in a single talk; a full course spread over a year or two would be required for such a project. So what I propose to

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do is to give my impressionistic view of this beautiful subject that lies between mathematics and physics.

It is exactly 60 years since the publication in 1932 of von Neumann's great book, *Mathematische Grundlagen der Quantenmechanik*. This book, which, in my opinion, is a landmark of twentieth century science, made it possible for the first time to speak about the new quantum mechanics in a mathematically blemishless manner. The ideas introduced in it provided the background for all subsequent developments in the mathematical aspects of quantum theory. During these years a number of new ideas and themes have been discovered, as a result of which the subject has been raised to an entirely new level: ideas of Weyl, Yang, Mills, and others regarding gauge theories and their relationship to the differential geometry of fiber bundles; ideas of Connes, Drinfel'd, Jimbo, Manin, and others concerning the theory of noncommutative spaces and their quantum groups of automorphisms; the work of Polyakov, Manin, Witten, and others, which has revealed profound links between field theory and complex algebraic and differential geometry; as well as the work of Weyl, Wigner, Mackey, and many others which had led to a profound understanding of the role of symmetry in quantum theory. For obvious reasons I shall confine myself only to a brief discussion of just a few of these, although even this is a very vast and difficult undertaking; in particular, I shall say nothing about questions of symmetry. I am also well aware that much of what I am going to say will be quite familiar to this audience. Nevertheless I hope you will get some pleasure out of this. It is like listening one more time to a piece of great music which never tires us no matter how often we have heard it before (even when the conductor has many obvious limitations).

2. THE RISE OF QUANTUM MECHANICS

The discovery and development of quantum mechanics during the early decades of this century is rightly regarded as a monumental achievement in the history of science and scientific thought. The ideas of Bohr, Heisenberg, Dirac, Schrödinger, Pauli, Born, and others helped create a new mechanics that differed in profound and revolutionary ways from classical mechanics, and which succeeded in giving a logically consistent description of atomic systems that had great predictive power. The novelty of its structure and the strangeness of its rules of physical interpretation immediately attracted the attention of mathematicians and mathematical physicists. One of the most interesting aspects of this new mechanics was that from the beginning it made contact with the new mathematics of the twentieth century that was simultaneously being developed. Up to that time all physical theories were firmly grounded in the classical mathematics of the nineteenth century, especially classical analysis and geometry; however, it was very clear from the beginning that quantum mechanics had to be formulated in the mathematical framework of linear spaces and linear transformations between them, and, when it became essential to inquire into the symmetry properties of quantum mechanical systems, of the theory of groups and their representations.

The new mechanics had many remarkable features, but none more so than the following:

Noncommutative algebra. The physical observables were represented as (the real) elements of a noncommutative algebra (over the complex number field and admitting an involution).

Classical limit. The kinematical and dynamical equations contained a parameter $\hbar$, Planck's constant; in the limit when $\hbar$ went to 0, these equations became the equations giving a classical description of the system in question.

Statistical interpretation. The principle, fundamental in all of classical physics, that in a given state all physical observables should possess sharply defined values, was abandoned and replaced by the weaker one that only the probability distributions of the observables could be calculated, even in principle.

Causality. Causality, which appeared to be lost because of the above statistical view of nature, was restored by saying that the quantum state, which was now identified with the totality of all statistical distributions of all the observables at a given instant of time, evolved in a causal and deterministic manner so that the statistics of observables changed in a causal manner, *as long as no measurements were made on the system.* But causality was lost when measurements were made; the dynamical evolution suffered a discontinuity whose result was not predictable in advance.

The second of these, namely the idea of using the smallness of Planck's constant $\hbar$ in comparing the new mechanics with the old one, was already a prominent feature of the Bohr (1913) theory of "stationary orbits" as the principle that for large quantum numbers the quantum orbits may be understood in classical terms, and was the forerunner of his great *correspondence principle.* Bohr of course was well aware that his stationary orbits were anomolous objects; indeed, they stood in complete contradiction to classical electrodynamics by their very definition, namely that they were the orbits in which an accelerating electron was not emitting or absorbing any radiation. Nevertheless, his deep understanding of the classical theory of electrons had led him to the conviction that only a radical break from it would have a chance of explaining the stability of atoms. His revolutionary ideas were still not sufficient and offered only a partial synthesis; but they highlighted the need for a more fundamental way of treating the atomic systems. He himself was a great force during this intermediate period from 1912 to 1925 in the search for the new mechanics, and it was his relentless insistence on a logically consistent description of atomic theory that ultimately paved the way for the epoch-making ideas of Heisenberg and the path to full understanding.

At the outset, following the appearance of the papers of Heisenberg (1925) and those of his collaborators (Born and Jordan, 1925; Born *et al.*, 1926) and of Dirac (1926), only elementary systems were treated, such as the hydrogen atom (Pauli, 1926).[1,*] But as the new principles became better understood, it became clear how they could be modified to include spin, to treat many-electron atoms, and even to satisfy the principle of special relativity.

3. MATHEMATICAL FOUNDATIONS OF QUANTUM MECHANICS

However, it was not until the appearance of von Neumann's book and papers that one could say that a proper and fully adequate mathematical foundation was available for the new theory to build on. He first of all gave an axiomatic treatment of (separable) Hilbert spaces, which up to that time were always just spaces of square summable functions. He then developed (independently of M. H. Stone, who had obtained essentially the same results) the spectral theory of unbounded operators, discovering in the process the precise class of operators which admitted spectral resolutions, thereby creating a far-reaching extension of Hilbert's spectral theory to unbounded operators. The foundational significance of this work resides of course in the fact that in the correspondence between physical observables and self-adjoint operators, the spectral resolution of the operator is the key ingredient, since it is from the resolution alone that one can calculate the statistics of the physical observable in various states. As long as only bounded observables are involved, it is obvious that one could dispense with operators and work with matrices, but, as soon as unbounded observables entered the picture, the matrix point of view became ambiguous and hopelessly inadequate, since the matrices could determine neither the operators nor their all-important spectral resolutions. This point is of essential importance, since the commutation rules of Heisenberg cannot be satisfied by bounded operators.[2]

The novel and revolutionary nature of the new mechanics brought in its wake many difficult problems of physical interpretation,[3] and von Neumann made a penetrating analysis of some of these questions in his

*Due to the length of some of the notes, all numbered footnotes are collected in a section at the end of this paper.

book. These problems have not been completely resolved; indeed, some of them are still being argued about, even today, even in this conference.

4. OUTLOOK

If one looks beyond these questions of interpretation and philosophy, and examine the success of quantum mechanics in explaining the microscopic world and predicting its behavior, one has to admit that it has been spectacularly successful. It was only when attempts were made to extend the scope of its framework to include quantum electrodynamics, i.e., a relativistic quantum theory of the electromagnetic field, that difficulties were encountered. These difficulties, some of which stem at least partially from the many unresolved problems of electromagnetic theory even at the classical level, have been eliminated for the moment by the concept of renormalization. As this is not part of the foundations of the structure erected already, one cannot say that everyone has been persuaded to accept the quantum theory of electromagnetic fields or its generalizations in their present form as definitive.[4] Nevertheless the theory of quantized fields as it exists at present is a structure of great beauty and heuristic content, with profound applicability to the real world as well as to the world of mathematics. One might therefore say that the theory of elementary particles and their interactions is in the form of a great unfinished symphony; one has a good understanding of many parts of it and has a conviction that a large part of reality has been discovered, but one has also the feeling that some pieces are still missing, perhaps new themes, or new ways of understanding old themes.

In this prolonged hiatus the attempts to overcome the difficulties of formulating a coherent and unified theory of elementary particles have led the physicists very deeply into present-day mathematics. A new generation of mathematicians and physicists has arisen, at home in the themes and languages of both disciplines, and their efforts to construct very diverse models of field theory and dynamics of infinite-dimensional systems have utilized to the utmost the resources of modern algebra and algebraic geometry. The concepts and results that have come out of their programs are already of great interest even from the purely mathematical standpoint in such areas as noncommutative geometry, holomorphic vector bundles, string theory and its relation to the theory of moduli, conformal field theory, and so on.[5] It is therefore very clear that an intriguing world of ideas that unites mathematics and physics at their deepest levels is being created at this time, and while it is impossible to predict either the success of these new attempts or the ultimate shape of a new view of nature that they may lead to, it is equally impossible not to feel a sense of real excitement.

5. ALGEBRAS OF OBSERVABLES AND GEOMETRIES OF PROPOSITIONS

Heisenberg and Dirac had already taken, explicitly and unambiguously, the great leap forward in asserting that in quantum theory the observables form a noncommutative algebra. But it was von Neumann who introduced the (partially ordered and orthocomplemented) set of experimentally verifiable propositions—the *logic* of quantum mechanics—as an object of equally fundamental significance. His point of view was that the quantum logic could in fact be taken as the basic object; its structure as a projective geometry contrasted sharply with the Boolean algebra occurring in classical mechanics, and it led to the algebra of observables as the algebraic object that *coordinatized* this geometry. He was the first person to consider geometries more general than the classical projective geometries, of possibly infinite dimension and even continuous in the sense there are no *points* = *minimal elements* (or "pointless geometries," as he is supposed to have often joked), and prove coordinatization theorems about such geometries. In my opinion these theorems and their successors, like Piron's theorem on standard logics, are among the most remarkable results in the foundations of the subject because they show that any nontrivial model has to look very much like the standard model.[6] For a general geometry the theorem of coordinates in the simplest situation constructs a vector space over a division ring whose linear subspaces reconstruct the given geometry; in more complicated cases the vector space disappears but we have an algebra whose lattice of principal left (or right) ideals is isomorphic to the geometry. The vector space reappears, however, if we consider the representations of this algebra! The algebra itself acquires some positivity properties coming from the probabalistic interpretation. States are viewed as linear forms of positive type on the algebra—the value of the linear form at an element of the algebra is the expectation value of the corresponding observable in the state represented by the linear form. Von Neumann's theory of operator algebras may thus be seen as an effort to construct and study analytically infinite-dimensional algebras of operators on a Hilbert space that are not standard, while his work on geometries may be viewed as exploring the same theme geometrically.[7]

I have passed over a little too quickly the relationship between states and representations. Given a complex algebra with an involution and a state on it, it was discovered in the early 1950s by many people (Gel'fand, Naimark, Godement, Segal, etc.) that there is a canonically associated Hilbert space, a $*$-representation of the algebra, and a unit vector in the Hilbert space such that the state is a multiple of the vector state defined by this unit vector. In this way, if one starts with a $*$-algebra whose real elements represent the physical quantities and introduce the states as the positive linear forms on the algebra, the Hilbert spaces and the vector states are just around the corner. This algebraic approach to quantum mechanics was systematically explored by Segal.[8]

This entire circle of ideas—geometries with or without points but typically orthocomplemented, their coordinatizing algebras acquiring an involutive structure originating from the orthocomplementation, the notion of states as positive linear forms that give rise to representations in Hilbert spaces—is a very beautiful and tightly knit one and we owe it to von Neumann that it is nowadays placed (casually, as it were) as the cornerstone of the mathematical foundations of the theory. This line of thought expressed in the chain

geometry → algebra → coordinatization → state → representation

informs his entire work and gives it a tremendous coherence.

This is not the place to enter into a discussion of how one can give a very satisfying treatment of most aspects of quantum mechanics starting from these principles (see, e.g., Dirac, 1958; Weyl, 1950). I do want, however, to mention briefly two things. The first concerns the probabilistic aspects of the foundations, about which I have said very little so far. Von Neumann had already realized that the calculus of traces on his operator algebras was the counterpart to the theory of integrals on Boolean algebras and hence may be viewed as examples of *noncommutative integration*. The operator algebraic impact of these ideas was immediate, but the probabilists waited for several decades before taking up this theme. I am thinking of the subject that is nowadays called quantum probability and studied by Accardi, Hudson, Meyer, Parthasarathy, Streater, and others.[9]

The second point is that I have implicitly assumed that the division ring that occurs in the coordinatization of the geometry of propositions is the complex number field. Strictly speaking, this is an additional assumption, as the division ring is an important invariant of the geometry. The standard model makes this assumption, but there are other possibilities.[10]

6. GLEASON'S THEOREM AND ITS GENERALIZATIONS

Belonging to this same circle of ideas but very different in its motivation and scope is the collection of results consisting of Gleason's theorem and its generalizations. This line of thought was first introduced by Mackey when he began his foundational analysis of quantum theory in the late 1950s. His basic idea was that in asserting that all the statistical aspects of a quantum system were contained in the expectation functional which was supposed to be a linear function on the algebra of observables, von Neumann had perhaps assumed too much, since the additivity of the expectation for noncommuting operators was, strictly speaking, not easily understandable even at a heuristic level. The remedy is of course to introduce the *physical states* in whose definition the additivity is required only for commuting operators. Von Neumann's work on traces in operator algebras has already shown that the additivity of the trace for noncommuting operators was true but a technically difficult thing to establish. The obvious question now is whether a physical state is necessarily a state in the von Neumann sense, i.e., whether a physical state is additive even for noncommuting operators. Mackey formulated this as the statement that all probability measures on the standard logic are von Neumann states and are thus determined in the usual manner by uniquely defined statistical operators (namely, operators U which are positive and have trace 1, the so-called *density matrices*). This was proved by Gleason under the caveat that the underlying Hilbert space has dimension at least 3.[11] Gleason's theorem is therefore at the heart of the fundamental development of the subject and it is not surprising that people have tried to extend it to other logics. It is my understanding that this has now been established for all logics arising from von Neumann algebras[12] (with appropriate restrictions of course).

Time does not permit me to discuss the questions of physical interpretation that have attracted enormous attention ever since the birth of quantum mechanics. I do, however, wish to point out that the results of von Neumann, Mackey, Gleason, and their successors have pretty much closed the door for the discovery of any nontrivial situation violating the canonical interpretations except in dimension two.[13] Of course it is also in dimension two that one encounters nonclassical geometries: the non-Desarguesian geometries, which are different from projective planes over division rings. Any theory of exceptional models, namely models in which accepted rules of definition and interpretation are not valid, must, I feel, start from these nonstandard geometries and the nonassociative structures related to them.

7. QUANTIZATION AS DEFORMATION

I now come to the second remarkable feature of quantum mechanics, namely the idea that although it is very difficult in its structure from classical mechanics, it should have classical mechanics as its limiting form when $\hbar$, Planck's constant, goes to 0. Clearly this is a requirement that is not at all easy to formulate in mathematical terms, since the two theories take place in entirely different settings. However, things are not as bad as they seem and the idea of quantization contains the link between the two theories.

As everyone knows, quantization is a process that provides the mechanism to associate to a given classical system a quantum system that *corresponds* to it or *quantizes it*. For a classical system with finitely many degrees of freedom the dynamics takes place in the phase space $\mathbf{R}^{2N}$ whose coordinates are the $q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N$ which are the positions and momenta; for the quantized system the action is in the Hilbert space $L^2(\mathbf{R}^N)$ of square summable functions of the *classical* coordinates $q_1, q_2, \ldots, q_N$; the classical Hamiltonian

$$H(p_1, p_2, \ldots, p_N, q_1, q_2, \ldots, q_N)$$

is then replaced by the same expression which represents the operator in the Hilbert space $L^2(\mathbf{R}^N)$ obtained by interpreting q_j as the operator of multiplication by q_j and p_j as the operator $-i\hbar\partial/\partial q_j$. This is at best an ambiguous prescription, since, even for polynomial H, the operator is not uniquely determined, due to the noncommutivity of the p's with the q's. Thus there is no unique way to quantize a classical system, although in very simple cases there is generally little doubt as to how to do the quantization. One of the very few general methods that is available is the so-called *Weyl quantization* proposed by Hermann Weyl.[14] I do not wish to go into details, but simply recall that this is a map $W: f \mapsto W(f)$ from functions f on the phase space $\mathbf{R}^{2N}$ to operators on the Hilbert space $L^2(\mathbf{R}^N)$. Here $W(f)$ is an integral operator for rapidly decreasing f, and for f of moderate growth, $W(f)$, although not an integral operator, is well defined on the Schwartz subspace of $L^2(\mathbf{R}^N)$ consisting of the rapidly decreasing functions of the q's and leaves it invariant; for instance, if f is a polynomial, $W(f)$ is a differential operator, obtained by replacing p_r, q_r respectively by $-i\hbar\partial/\partial q_r, q_r$, with the understanding that each monomial in the p's and q's is replaced by the average of the corresponding operator monomial expressions over all possible orderings. The map W is one to one. Weyl's quantization allows one to take, for example, any real polynomial classical Hamiltonian and associate to it a quantum Hamiltonian.

Once the two settings are related to each other in this manner, it is possible to formulate the idea that the quantum dynamics tends to the classical dynamics as $\hbar$ tends to 0. The most successful of such formulations is the one that says that the *entire quantum algebra becomes, in the classical limit, the classical algebra of functions on the phase space, the algebraic structure being the one furnished by the Poisson bracket.* The simplest way to do this is to use the Weyl quantization map to pull back the quantum algebra to the classical function algebra and investigate all limiting processes in the latter. Weyl did not do this; this was done by Moyal many years later.[15] Moyal found that if one transfers the Lie algebra structure of the space of operators of the Hilbert space to the space of functions on phase space by the Weyl quantization map, one obtains not the classical Poisson bracket but a *deformation of it*, namely, a bracket structure *which* *tends to the Poisson bracket as $\hbar \to 0$.* This was the first clear-cut result that showed that quantum mechanics, represented by the Lie algebra of operators on $L^2(\mathbf{R}^N)$, was a *deformation* of classical mechanics represented by the algebra of smooth functions on the phase space of p's and q's, viewed as a Lie algebra with respect to Poisson bracket. In this way a connection was made with the prophetic idea of Dirac expressed in his very first paper[16] that the commutator bracket of operators in quantum mechanics is *analogous* to the classical Poisson bracket. The Poisson structure induced by the Weyl quantization map on the space of functions on the classical phase space is called the *Moyal bracket* nowadays.

In this direction of thought, the obvious question at this stage is to ask whether quantum mechanics furnishes the *only possible* deformation of the classical Poisson algebra, namely, $C^\infty(\mathbf{R}^{2N})$ viewed as a Lie algebra with respect to the Poisson bracket. To my knowledge this converse question was first studied by Flato, Lichnerowicz, Vey, and their collaborators (Sternheimer, Fronsdal, Bayen, Gutt, Lecomte, etc.). It followed from their work that the Moyal bracket was, up to certain natural notions of equivalence, the only possible deformation of the classical Poisson algebra within a wide range of possibilities. They did this by studying general deformations of arbitrary Lie algebras and applying the results obtained to the classical Poisson algebra. Their methods went much farther and clarified the whole question of quantization of classical systems in very general contexts, such as in curved space or in the presence of constraints, and showed that under certain very general conditions (vanishing of certain cohomology groups) the Moyal deformation is the only mechanism available for quantization of a classical system on an arbitrary *symplectic* manifold.[17]

I do not have to overemphasize the foundational significance of these deformation-theoretic results. They show that once we build the correspondence principle as an integral part of the theory and relate it in a natural manner to the observable algebra, the mathematical structure of the theory is essentially uniquely determined.

8. NONCOMMUTATIVE GEOMETRY. QUANTUM GROUPS

The discussion up to this point has led to the view that quantization is a deformation of commutative associative algebras changing them to noncommutative associative algebras, and that there is an essentially unique way of doing this. In the last few years this theme has been revived in a big way, and the chain of thought

$$\textit{commutative algebra} \to \textit{deformation} \to \textit{noncommutative algebra}$$

has been explored in a very systematic manner. The impulses for the new progress came from several directions: from Connes, whose profound understanding of von Neumann algebras led him to go beyond measure theory and topology in the noncommutative domain and take the big step into noncommutative differential geometry,[18] from the Leningrad school of physicists led by Faddeev, whose work on the quantum inverse problem led them to new structures that could be understood as deformations of classical Lie groups in the noncommutative realm,[19] and finally from Drinfel'd, Jimbo, Manin, and others, who realized that these deformations of the classical Lie groups are really to be understood as *quantum groups*.[20]

I think the beginning of the chain of thought that is at the heart of these new developments lies in the work in the 1930s due to Stone and Gel'fand. Stone proved that an abstract Boolean algebra is isomorphic to the Boolean algebra of the open and closed subsets of a compact space canonically associated to it, while Gel'fand proved that a commutative Banach algebra (under certain regularity conditions that are anyway essential) was isomorphic to the algebra of continuous functions on a compact space canonically associated to it. This idea was then taken up in algebraic geometry by Grothendieck in the late 1950s. Grothendieck's point of view was to insist that every commutative ring has to be viewed as the ring of regular functions on its spectrum, and that algebraic geometry was the study of spaces built locally like these "spectra" (*schemes*). The total fusion between algebra, geometry, and arithmetic brought about by Grothendieck's view led to the new revolution in algebraic geometry in which the leading event was the monumental work of Grothendieck himself. Thus the geometry of the spaces was a total reflection of the ring of functions on them. The idea of Connes, Drinfel'd, Jimbo, Manin, and others is to take the Grothendieck approach to the next stage by extending it to noncommutative rings, and view them as the "functions on a noncommutative space." For example, for any complex number $\hbar$ (Planck's constant!) the $\mathbf{C}$-algebra $\mathbf{C}_\hbar[x, y]$ with two generators x, y, and a single relation

$$yx = e^{\hbar}xy$$

is to be thought of as the ring of functions on the quantum plane $\mathbf{C}_\hbar^2$, which tends to the classical plane when $\hbar \to 0$. In this sense therefore quantization is a part of the chain of thought

commutative geometry → noncommutative geometry

This heuristic philosophy suggests also that one should think of the *quantum groups* as groups of automorphisms of the noncommutative spaces. This is then a question of studying the deformations of the automorphism groups of the commutative structures in a systematic manner.

Let me now briefly indicate how this is done. As I mentioned above, there is in general no distinction in modern algebraic geometry between an algebraic variety and the algebra of regular functions on it. If the variety is an algebraic group, say G, then we put this in evidence by introducing the maps defining multiplication and inverse,

$$G \times G \to G, \qquad G \to G$$

which give rise to the dual maps involving the algebra $F(G)$ of all regular functions on G,

$$\Delta\colon \quad F(G) \to F(G) \otimes F(G), \qquad S\colon \quad F(G) \to F(G)$$

The associativity law, the existence of a unit element, and an inverse can all be interpreted via these dual maps and what we get is a *Hopf algebra* structure for $F(G)$. This Hopf algebra is commutative, but, and this is the key point, it is allowed to become noncommutative under deformation. In this way we arrive at the new point of view: the Hopf algebras are to be the *quantum groups*. One can also reach the same level of understanding by working in the dual context where the function ring $F(G)$ is replaced by the enveloping algebra $U(G)$ of the Lie group G. $U(G)$ is a Hopf algebra which is in duality with $F(G)$, and so is cocommutative rather than commutative [cocommutativity of $U(G)$ means that Δ commutes with the flip $a \otimes b \to b \otimes a$ on $U(G) \otimes U(G)$]. It is reasonable to expect that the Hopf algebras (which are now not required to be cocommutative) obtained by deforming the $U(G)$ will be in duality with the ones that are obtained by deforming the $F(G)$ and so are equally entitled to be viewed as quantum groups. The great discovery of Drinfel'd and Jimbo was that while an algebraic or a Lie group may be rigid within the category of groups—this is the case for the simple Lie groups—it may be deformable in a nontrivial manner *as a quantum group*. More precisely, its function ring (or equivalently, its enveloping algebra) may have nontrivial deformations in the category of Hopf algebras (see references in note 20).

These ideas have attracted a great deal of attention and led to an immense amount of activity and results. At the same time questions are also being raised as to the relevance of these ideas to the great unresolved problems of quantum field theory of elementary particles and their interactions. I feel that these questions and doubts cannot be answered as yet. The most pragmatic as well as philosophically sound stance to take is that these new themes have created new mathematical structures that allow new models of space-time and their symmetry groups to be built, and that it is not unreasonable to hope, or at least speculate, that these quantum spaces and quantum groups may have a role to play in a theory of space-time dynamical processes of elementary particles that is free of divergences.

In spite of the very sketchy nature of my treatment, I hope I have conveyed to the reader the point of view that noncommutative geometry is just another step in the long march leading from commutativity to noncommutativity that has been inspired by quantum theory.

9. GAUGE THEORIES

I finally turn to an entirely different theme, namely that of gauge theory. Gauge theories go back to Hermann Weyl and his attempts starting during 1918–1921 to unify gravitation and electromagnetism (Weyl, 1968, Vol. II, pp. 29, 55). These attempts failed because of difficulties of interpretation,[21] but he revived the idea behind them (which he must have found too beautiful to abandon) in the late 1920s in a quantum-theoretic context.[22] It was also at about the same time that Dirac's famous paper on magnetic monopoles appeared (Dirac, 1931). The fundamental idea behind these papers was that the quantum-theoretic wave function of an electron could only be defined locally and one had the freedom of multiplying it by a phase factor that was space-time dependent; the path-dependent scale factor of Weyl's attempted unification of gravity and electromagnetism had now become the path-dependent phase factor characteristic of electromagnetism in the quantum domain! The freedom of multiplication by the phase factor of course did not change the probability distribution of the location of the electron, but led to new possibilities; monopoles, for instance, quantization of electric charge, and so on. In mathematical terms, the Hilbert space of functions on a manifold was replaced by the Hilbert space of sections of a Hermitian line bundle on the manifold. Furthermore, the transport of the ambiguous phase factors along paths in space or space-time meant that there were connections on this line bundle; the (local) components of the 1-forms of these connections were identified with the vector potentials of electromagnetism, and their curvature identified with the electromagnetic tensor. But the full scope of this circle of ideas was not realized until Yang and Mills (1954) discovered their famous equation and established its invariance under the full infinite-dimensional group of gauge automorphisms of the bundle. Unlike the case of electromagnetism, the bundle was now of rank 2 and the potentials were $SU(2)$ connections, and the Yang–Mills equation,[23] a generalization of Maxwell's equation, became nonlinear. I must mention here that as a consequence of these discoveries, the vector potentials have acquired a physical significance of their own. The Bohm–Aharonov–Chambers experiment has shown unmistakably that there may be nontrivial phenomena involving the phase of the electron even in regions where the electromagnetic field is zero, and the formal structure of the Yang–Mills equation implies very clearly that it is the potentials that satisfy the gauge-invariant equation, not the field strengths.[24] It is now universally accepted that elementary particle theories have to be gauge theories, the structure group of the bundles in question being a reflection of the internal symmetry groups of the particles. The interaction between physics and the geometry of fiber bundles generated by these ideas has led to profound results in both fields. Time does not allow me to go into more detail.[25]

The same line of thought that the wave function is not a function but a section of a Hermitian line bundle is also at the basis of the more recent ideas on quantizing systems of several identical particles (Finkelstein and Rubinstein, 1968; Leinass and Myrheim, 1977; Wilczek, 1990), but with a statistics that is different from the usual Fermi or Bose statistics. This is the theory of the so-called anyons with their fractional statistics, and the corresponding theories of quantized two-dimensional systems have many interesting features that make them suitable objects of study.

10. INTERPRETATION AND MEASUREMENT

Due to lack of time I have been unable to even touch on the very important themes of interpretation and measurement in quantum theory. Interest in these aspects has increased tremendously in view of the beautiful experiments that have been performed in recent years, such as traps for individual particles, neutron interferometer experiments, photons widely separated in space-time, and so on. To develop a consistent theory of these and other experiments that affect fundamental physics is an important problem to which increasing attention is bound to be given in the coming years. It appears to me that there is a need for a theory of measurements that takes into account aspects involving dynamics as well as covariance with respect to the Galilei group or the Poincaré group.

11. CONCLUSION

No one is more aware than myself of the very incomplete nature of the discussion I have given here, although I have made an attempt to compensate by giving a number of references to the literature. However, my only aim has been to look along with you at a beautiful but constantly changing picture and point out some marvelous features that have fascinated me for a long time. If I have communicated even a fraction of the excitement that I feel in thinking about these ideas, I have fulfilled the task I set out for myself.

NOTES

[1]An excellent source for all the above papers is Van der Waerden (1968), which reprints them with annotations and historical analysis. Also, for the general historical development of quantum mechanics I refer the reader to the monumental volumes of Mehra and Rechenberg (1982).

[2]The point is that, up to unitary equivalence, p and q must be represented by the operators [in $L^2(-\infty, +\infty)$]

$$f(q) \mapsto -i\hbar \frac{d}{dq} f(q), \qquad f(q) \mapsto q f(q)$$

respectively, and these are obviously unbounded operators. This of course follows from the well-known theorem of Stone and von Neumann that if we start with the commutation rule between p and q expressed in (Weyl's) integrated form, then the above representation (where f can be vector-valued) is the only possible one up to unitary isomorphism.

[3]The best starting point for reading about the problems of physical interpretation and their resolution is Bohr's (1969) article summarizing the so-called Bohr–Einstein dialogues. After the Solvay conference of 1930, Einstein no longer questioned (publicly at least) the *consistency* of the quantum mechanical interpretation (the so-called Copenhagen interpretation); but his attitude toward the *completeness* of the quantum mechanical description of atomic systems was an entirely different matter. Einstein formulated his objections in a dramatic manner in the famous paper with Podolsky and Rosen entitled "Can quantum mechanical description of physical reality be considered complete?" (Einstein *et al.*, 1935). These were answered by Bohr (1935) in an equally famous paper with exactly the same title. For an absorbing account of this episode the reader may consult Rosenfeld (1967). Both the papers and the relevant excerpt from Rosenfeld's article are reprinted in the monumental work edited by Wheeler and Zurek (1983, pp. 138–151). This book contains reprints of the Bohr–Einstein dialogues mentioned above, includes a reprint of Chapters V and VI of von Neumann's book that contain his discussion on the measurement problem, and contains as well as a number of articles that relate to the whole question of measurement in quantum theory.

Von Neumann's name is always attached to two specific but interrelated questions:

1. Are there "hidden variables" in the quantum mechanical description of nature which give rise to its statistical character?
2. Is there a way to comprehend under a single unified scheme the two widely different ways in which the quantum state changes, namely, the continuous evolution on the one hand determined by solving (in principle) the Schrödinger equation, and the discontinuous change that occurs when measurements are made?

I can do no more than give a brief glimpse of the results of his beautiful and highly original treatment of these questions and refer the reader to his book (von Neumann, 1932; 1955), or at least to the summary I have given in Varadarajan (1985) for more detail. I must urge everyone, especially young people, who are starting to get interested in these problems to read von Neumann's argumentation, not just for the mathematics, which is very interesting but not really difficult at all, but for the coherence and eloquence as well as the conciseness with which it is presented.

Von Neumann's answer was a "no" to the first question (see Chapter IV of his book) and a "yes" to the second (Chapters V and VI). For the first his starting point was to identify the statistical ensembles with the states (or the physical states, see below) of the observable algebra. They form a convex set, and if there are any dispersion-free states, they must be among the extreme points of this convex set. He then proved that the extreme points are precisely the vector states $A \mapsto (A\varphi, \varphi)$, and established easily that these were never dispersion-free. The second question of course requires a preliminary analysis of the changes in the states brought about by the measuring process. He proved that any measurement induces a change of state which is an endomorphism of the convex set of states with the striking property that it decreases the "purity" of the state. More precisely, it can change pure states into mixed states, but can never do the opposite. He realized that this is a phenomenon of a thermodynamic nature and showed that the *entropy* $-\operatorname{tr}(U \log U)$ of the mixed state represented by the statistical operator U *always increases* when measurements are made, and so the changes wrought by the measurement process are irreversible. This done, he viewed the measurement process on a system I as the quantum theory of the system I + II, where II represents the system composed of the measurement apparatus, the observer, etc. In the system I + II the state of course evolves causally according to the Schrödinger equation and the changes are reversible, but the changes of state induced in the system I by this evolution are irreversible and thermodynamic. He then showed, using a plausible scheme that formalizes the notion of measurement of a quantity, that the discontinuous change of state in I is identical with that induced by the causal change of state in I + II. For references to some of the many papers and results that have since touched on this question, see Wheeler and Zurek (1983); for a systematic exposition of the quantum theory of measurement see the recent monograph of Busch *et al.* (1991).

[4]See, for example, Dirac's comments on quantum field theory (Dirac, 1966, p. 2; 1958, pp. 309–310).

[5]For the references to noncommutative geometry see below; for the relationship between the theory of holomorphic vector bundles and gauge theories (instantons, etc.) see Manin (1988*a*); for recent discoveries concerning particle physics, string theories, and their mathematical formalism, see the general survey article of Witten (1987). For further discussions on string theory and complex algebraic geometry, see Manin (1987).

[6]According to I. Shafaraevich, the "unpronounceable" word "coordinatization" was coined by Hermann Weyl. Shafaraevich places this concept at the heart of algebra and uses it as the guiding thread in his beautiful book, *Algebra* (Shafaraevich, 1990). In some sense this duality between the algebras and the spaces they coordinatize is also the central theme of my talk, and will reappear at various places in what follows.

The classical projective geometries are described usually in terms of their incidence properties. Attention is generally focused on the plane and three-dimensional space and so the elements of the geometry are divided into points, lines, and planes, and their incidence properties are exactly the same as the ones all of us learn when we first study geometry. But this becomes cumbersome in higher dimensions and the best way to proceed is to follow Birkhoff's idea that we are dealing with a modular complemented irreducible lattice of finite rank; if the rank is ≥ 4, then it is isomorphic to the lattice of linear subspaces of a vector space over a division ring of dimension equal to the rank of the geometry. For a plane geometry, i.e., a geometry of rank 3, this is not true unless Desargues' theorem on perspective triangles is assumed to be true in the geometry. In their famous paper, "The logic of quantum mechanics," Birkhoff and von Neumann (1936) proved that orthocomplementations in a projective geometry are precisely the ones that arise in a natural manner from definite scalar products on the vector space.

Von Neumann realized that one should try to prove such coordinatization theorems for geometries with infinite rank and possibly even without points. His book, *Continuous Geometry*, published posthumously (von Neumann, 1960) contains one of the most far-reaching theorems of this type. This theorem asserts that any lattice which is modular, complemented, and has rank $N \geq 4$ is isomorphic to the lattice of principal right ideals of the ring of $N \times N$ matrices from a regular ring (the matrix ring is then also regular), and that orthocomplementations in the geometry correspond in a natural manner to definite involutions of the ring. Here rank is defined in such a manner that it does not imply the existence of points, and regularity is a (necessary) technical condition. For geometries without the modular axiom the results are less definitive as far as I am aware. If points are assumed to exist and if it is also assumed that the elements contained in a given element form a projective geometry, then it was proved by me (Varadarajan, 1985) that the geometry is that of all finite-dimensional subspaces of a vector space over a division ring. This theorem can then be used as the foundation for obtaining a proof of Piron's beautiful and fundamental characterization of standard logics as complete projective logics (Piron, 1964, p. 439; see also my book above, pp. 114–122).

Another approach to the foundations starting from the measurement algebra was worked out in a series of papers in the *Proceedings of the National Academy of Sciences USA* in the 1960s by J. Schwinger. These were then published as a book (Schwinger, 1970). These papers are very thought provoking and the mathematician should take a serious look at them. Last but not least, the book of Mackey (1963) reexamined the von Neumann synthesis and pushed it forward in a profound manner.

[7]One of the great motivations behind von Neumann's monumental work on operator algebras was that they provided many nontrivial examples of logics other than the standard ones. He discovered, for instance, that the projections in a factor of type II_1 form a continuous geometry, and that if we enlarge the factor by including all the operators *affiliated to the factor*, namely those left invariant by all the unitaries that fix all the elements of the factor, then we get exactly the regular ring whose right principal ideals correspond to the projections in the factor. Through new examples (von Neumann, 1961, Volume IV) he showed that there are continuous geometries that are not obtained from type II_1 factors as above.

[8]Over the years I.E. Segal has been a great force in insisting on an emphasis on fundamental issues in quantum mechanics, and anyone who is interested in understanding either the foundations or the difficult problems of quantum field theory should study his papers. For his treatment of the foundations where the observable algebra plays the central role, see Segal (1947).

[9]For a survey of noncommutative integration see Connes (1979, 1982). For quantum probability see Parathasarathy (1992). This is a very nice introduction to problems and ideas regarding what may be described as quantum diffusion.

[10]For quaternionic quantum mechanics the famous paper is of course that of Finkelstein *et al.* (1962). For a discussion of real quantum mechanics see Mackey (1978). This book also contains a number of rather profound discussions on fundamental aspects of quantum theory that are well worth studying carefully.

[11]Gleason's theorem was proved in Gleason (1957). Its assertion is that if p is a map of the set of projections of Hilbert space $\mathbf{H}$ of dimension ≥ 3 into the unit interval $[0, 1]$ which is a probability measure in the sense that (i) $p(0) = 0, p(I) = 1$ and (ii) p is countably additive over orthogonal projections, i.e.

$$p\left(\sum_j Q_j\right) = \sum_j p(Q_j), \qquad Q_j Q_k = Q_k Q_j = 0 \qquad \text{for} \quad j \neq k$$

then p is the measure determined by a von Neumann statistical operator, i.e., there is a unique operator U which is positive and has trace 1 such that

$$p(R) = \mathrm{tr}(RUR) \qquad \text{for all projections } R \text{ in } \mathbf{H}$$

From the point of view of foundations the importance of this result is that it allows one to replace "state" by "physical state" in the foundational discussions of von Neumann, thereby greatly increasing their scope; for instance, in the question of hidden variables.

[12]The question is that of determining all the probability measures on the lattice of projections of a von Neumann algebra of operators in a Hilbert space. This can be formulated as the question of first showing that physical states are states, and then representing states in terms of traces. This issue appears to be well understood at this time; see the references in Varadarajan (1985, pp. 146–147).

[13]The foundational questions are just too many to be treated with any justice in this talk. I want to remind the reader that measurement problems involving quantum *fields* are also very important and were first considered by Bohr and Rosenfeld (1933) [reprinted in Wheeler and Zurek (1983, p. 479). The reader would also wish to consult for many of these questions the book of Beltrametti and Cassinelli (1981), as well as the monograph of Busch *et al.* (1991).

[14]Weyl quantization was introduced in Weyl (1927). It was also discussed in Weyl (1950). It is done through the representation theory of the Heisenberg group. For details see Varadarajan (1991).

[15]The Moyal bracket on the Poisson Lie algebra of functions of p and q is defined as follows (Moyal, 1949). Let $\{f, g\}_\hbar$ be the bracket depending on the parameter $\hbar$ of two functions f, g, with $\{f, g\}_0$ as the Poisson bracket. Then

$$\{f, g\}_\hbar = \sum_{r \geq 0} \frac{(-\hbar^2/2^2)^r}{(2r+1)!} P^{2r+1}(f, g)$$

where P is the *bidifferential operator* acting on pairs of functions (u, v) on $\mathbf{R}^{2N}$ by

$$P(u, v) = \sum_r \left(\frac{\partial u}{\partial p_r} \frac{\partial v}{\partial q_r} - \frac{\partial u}{\partial q_r} \frac{\partial v}{\partial p_r} \right) = \{u, v\}_0$$

For the formal calculation starting from Weyl quantization, which is quite beautiful, see Varadarajan (1991).

[16]Dirac's (1926, Section 4) presentation makes it quite clear that the Lie bracket in the quantum algebra is the analog of the classical Poisson bracket, and goes over to the Poisson bracket in the limit when $\hbar \to 0$. See also the treatment in Dirac (1958, p. 85).

[17]For the theory of deformations of Poisson structures on symplectic manifolds see Flato and Sternheimer (1980) and the references given there. The uniqueness of the Moyal deformation is a local question. The existence requires some conditions, for example, Vey's $H^3(W, \mathbf{R}) = 0$, on the symplectic manifold W. It is of course satisfied in the Euclidean case.

[18]A. Connes has several papers and reports where he has given expositions of his views on noncommutative geometry as well as references to the work of others, such as Riefel, Effros, and so on. The most detailed of these is Connes (1985). See also Witten (1985).

[19]For a detailed review of the origins, history, and development of quantum groups one of the best places to start is Drinfel'd (1987).

[20]E.g., Jimbo (1985, 1986*a*–*c*). Manin's ideas are given a brilliant exposition in Manin (1988*b*, 1991). Manin considers the deformations of $F(G)$ rather than the dual $U(G)$ and constructs a general theory of such deformations when $G = GL(N)$. For my own work on deformations and quantum groups, see Truini and Varadarajan (1991, 1992, 1993).

[21]For a beautiful account of Weyl's theory and the objections of Einstein see Yang (1986).

[22]Yang (1986) also discusses the quantum version of the gauge theory of the electron that Weyl worked out in Weyl (1929) [reprinted in Weyl (1968, Vol. III, p. 245)].

[23]See also Yang (1983, p. 172). Yang's own very interesting commentary appears on p. 19 of Yang (1983).

[24]Wu and Yang (1974, 1975). See also Yang (1983, pp. 457, 460), commentary on p. 73). Wu and Yang (1975) is especially nice, giving a very comprehensible account of the Bohm–Aharonov–Chambers experiment, the interpretation of it through vector bundles, and the dictionary between gauge-theoretic concepts and concepts from the theory of fiber bundles. It is also interesting to read Yang (1980). There is finally the basic paper of Aharonov and Bohm (1959).

[25]For a beautiful account of classical gauge theories and the problems and results in the theory of holomorphic and Hermitian vector bundles see Atiyah (1979) [reprinted in Atiyah (1988)]. See also Manin (1988).

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